FIXED POINT THEOREMS OF ĆIRIĆ-MATKOWSKI TYPE IN GENERALIZED METRIC SPACES

MORTAZA ABTAHI

Abstract. A self-map T of a ν-generalized metric space (X, d) is said to be a Ćirić-Matkowski contraction if $d(Tx, Ty) < d(x, y)$, for $x \neq y$, and, for every $\epsilon > 0$, there is $\delta > 0$ such that $d(x, y) < \delta + \epsilon$ implies $d(Tx, Ty) \leq \epsilon$. In this paper, fixed point theorems for this kind of contractions of ν-generalized metric spaces, are presented. Then, by replacing the distance function $d(x, y)$ with functions of the form $m(x, y) = d(x, y) + \gamma\big(d(x, Tx) + d(y, Ty)\big)$, where $\gamma > 0$, results analogue to those due to P.D. Proiniv (Fixed point theorems in metric spaces, Nonlinear Anal. 46 (2006) 546–557) are obtained.

1. Introduction

Throughout the paper, the set of integers is denoted by $\mathbb{Z}$, the set of nonnegative integers is denoted by $\mathbb{Z}^+$, and the set of positive integers is denoted by $\mathbb{N}$.

Fixed point theory in metric spaces have many applications. It is natural that there have been several attempts to extend it to a more general setting. One of these generalizations was introduced by Branciari in 2000, where the triangle inequality was replaced by a so-called *quadrilateral inequality*. They introduced the concept of ν-generalized metric spaces as follows; see also [2, 5, 8, 15].

Definiton 1.1 (Branciari [3])**.** Let X be a nonvoid set and $d : X \times X \to [0, \infty)$ be a function. Let $\nu \in \mathbb{N}$. Then (X, d) is called a *ν-generalized metric space* if the following hold:

(1) $d(x, y) = 0$ if and only if $x = y$, for every $x, y \in X$;
(2) $d(x, y) = d(y, x)$, for every $x, y \in X$;
(3) $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$, for every set $\{x, u_1, \ldots, u_\nu, y\}$ of $\nu + 2$ elements of X that are all different.

Obviously, (X, d) is a metric space if and only if it is a 1-generalized metric space. In [2], the completeness of ν-generalized metric spaces are discussed. In [14], it is shown that not every generalized metric space has the compatible topology.

Definiton 1.2. Let (X, d) be a ν-generalized metric space. Let $k \in \mathbb{N}$. A sequence $\{x_n\}$ in X is said to be *k-Cauchy* if

$$\lim_{n\to\infty} \sup\{d(x_n, x_{n+1+mk}) : m \in \mathbb{Z}^+\} = 0. \tag{1.1}$$

The sequence $\{x_n\}$ is said to be *Cauchy* if it is 1-Cauchy.

The concept of Cauchy sequences in ν-generalized metric spaces are studied in [2, 15]; see also [3].

Date: June 1, 2015.

Proposition 1.3 ([2] and [15]). *Let (X, d) be a ν-generalized metric space and let $\{x_n\}$ be a sequence in X such that x_n $(n \in \mathbb{N})$ are all different. Suppose $\{x_n\}$ is ν-Cauchy. If ν is odd, or if ν is even and $d(x_n, x_{n+2}) \to 0$, then $\{x_n\}$ is Cauchy.*

A sequence $\{x_n\}$ in a ν-generalized metric space (X, d) is said to *converge* to x if $d(x, x_n) \to 0$ as $n \to \infty$. The sequence $\{x_n\}$ is said to *converge to x in the strong sense* if $\{x_n\}$ is Cauchy and $\{x_n\}$ converges to x. The space X is said to be *complete* if every Cauchy sequence in X converges.

Proposition 1.4 ([15]). *Let $\{x_n\}$ and $\{y_n\}$ be sequences in X that converge to x and y in the strong sense, respectively. Then*

$$d(x, y) = \lim_{n\to\infty} d(x_n, y_n).$$

Branciari, in [1], proved a generalization of the Banach contraction principle. As it is mentioned in [2], their proof is not correct because a ν-generalized metric space does not necessarily have the compatible topology; see [6], [12, 13, 14] and [16]. A proof of the Banach contraction principle, as well as proofs of Kannan's and Ćirić's fixed point theorems, in ν-generalized metric spaces, can be found in [15].

Theorem 1.5 ([15]). *Let X be a complete ν-generalized metric space, and let T be a self-map of X. For every $x, y \in X$, let*

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{1.2}$$

Assume there exists $r \in [0, 1)$ such that $d(Tx, Ty) \le rm(x, y)$, for all $x, y \in X$. Then T has a unique fixed point z and, moreover, for any $x \in X$, the Picard iterates $T^n x$ $(n \in \mathbb{N})$ converge to z in the strong sense.

The paper is organized as follows. In section 2, we study Cauchy sequences in ν-generalized metric spaces. We present a necessary and sufficient condition for a sequence to be Cauchy. Next, in section 3, we give new fixed point theorems in ν-generalized metric spaces. These results are generalizations to ν-generalized metric spaces of theorems of Meir and Keeler [10], Ćirić [4] and Matkowski [9, Theorem 1.5.1], and Proinov [11].

2. Results on Cauchy Sequences

The following is the main result of the section.

Lemma 2.1. *Let $\{x_n\}$ be a sequence in a ν-generalized metric space X such that x_n $(n \in \mathbb{N})$ are all different. Suppose, for every $\epsilon > 0$, for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$, if $\limsup_{i\to\infty} d(x_{p_i}, x_{q_i}) \le \epsilon$, then, for some N,*

$$d(x_{p_i+1}, x_{q_i+1}) \le \epsilon \quad (i \ge N). \tag{2.1}$$

If $d(x_n, x_{n+1}) \to 0$, then $\{x_n\}$ is ν-Cauchy.

Proof. Suppose $\{x_n\}$ is not ν-Cauchy. Then (1.1) fails to hold for $k = \nu$. Hence, there is $\epsilon > 0$ such that

$$\forall k \in \mathbb{N},\ \exists\, n \ge k, \quad \sup\{d(x_n, x_{n+1+m\nu}) : m \in \mathbb{Z}^+\} > \epsilon. \tag{2.2}$$

Since $d(x_n, x_{n+1}) \to 0$, there exist positive integers $k_1 < k_2 < \cdots$ such that

$$d(x_n, x_{n+1}) < \epsilon/i \quad (n \ge k_i).$$

For each k_i, by (2.2), there exist $n_i \geq k_i + 1$ and $m_i \in \mathbb{Z}^+$ such that

$$d(x_{n_i}, x_{n_i+1+m_i\nu}) > \epsilon.$$

Since $d(x_{n_i}, x_{n_i+1}) < \epsilon$, we have $m_i \geq 1$. We let m_i be the smallest number with this property so that $d(x_{n_i}, x_{n_i+1+m_i\nu-\nu}) \leq \epsilon$. Now, let $p_i = n_i - 1$ and $q_i = n_i + m_i\nu$. Then $q_i > p_i \geq k_i$, and

$$d(x_{p_i+1}, x_{q_i+1}) > \epsilon, \quad d(x_{p_i+1}, x_{q_i+1-\nu}) \leq \epsilon.$$

Using property (3) in Definition 1.1, since all x_n ($n \in \mathbb{N}$) are different, for every $i \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{p_i}, x_{q_i}) \leq d(x_{p_i}, x_{p_i+1}) &+ d(x_{p_i+1}, x_{q_i+1-\nu}) \\ &+ d(x_{q_i+1-\nu}, x_{q_i-\nu}) + \cdots + d(x_{q_i-1}, x_{q_i}). \end{aligned}$$

Therefore, $d(x_{p_i}, x_{q_i}) \leq \nu\epsilon/i + \epsilon$, and thus $\limsup_{i\to\infty} d(x_{p_i}, x_{q_i}) \leq \epsilon$. This is a contradiction, since $d(x_{p_i+1}, x_{q_i+1}) > \epsilon$, for all i. □

Theorem 2.2. *Suppose $\{x_n\}$ satisfies all conditions in Lemma* 2.1*, and, moreover, $d(x_n, x_{n+2}) \to 0$. Then $\{x_n\}$ is Cauchy.*

Proof. By Lemma 2.1, the sequence $\{x_n\}$ is ν-Cauchy. Since $d(x_n, x_{n+2}) \to 0$, by Proposition 1.3, the sequence $\{x_n\}$ is Cauchy. □

Theorem 2.3. *Let $\{x_n\}$ be a sequence in X such that x_n ($n \in \mathbb{N}$) are all different and $d(x_n, x_{n+1}) + d(x_n, x_{n+2}) \to 0$. Assume $m(x, y)$ is a nonnegative function on $X \times X$ such that, for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$,*

$$\limsup_{i\to\infty} m(x_{p_i}, x_{q_i}) \leq \limsup_{i\to\infty} d(x_{p_i}, x_{q_i}). \tag{2.3}$$

The following condition then implies that $\{x_n\}$ is Cauchy: for every $\epsilon > 0$, for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$, if $\limsup m(x_{p_i}, x_{q_i}) \leq \epsilon$, then, for some N,

$$d(x_{p_i+1}, x_{q_i+1}) \leq \epsilon \quad (i \geq N).$$

Proof. Follows directly from Lemma 2.1 and Theorem 2.3. □

3. Fixed Point Theorems of Ćirić-Matkowski Type

Let (X, d) be a ν-generalized metric space. A mapping $T : X \to X$ is said to be a *Ćirić-Matkowski contraction* if $d(Tx, Ty) < d(x, y)$, for every $x, y \in X$, with $x \neq y$, and, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, y \in X, \quad d(x, y) < \delta + \epsilon \Longrightarrow d(Tx, Ty) \leq \epsilon. \tag{3.1}$$

Lemma 3.1 ([1, Lemma 3.1])**.** *For a sequence $\{x_n\}$ in X and a nonnegative function $m(x, y)$ on $X \times X$, the following are equivalent:*

(i) *for every $\epsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{Z}^+$ such that*

$$\forall p, q \geq N, \quad m(x_p, x_q) < \epsilon + \delta \Longrightarrow d(x_{p+1}, x_{q+1}) \leq \epsilon. \tag{3.2}$$

(ii) *for every $\epsilon > 0$, for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$, if $\limsup m(x_{p_i}, x_{q_i}) \leq \epsilon$ then, for some N, $d(x_{p_i+1}, x_{q_i+1}) \leq \epsilon$ ($i \geq N$).*

Now, suppose T is a Ćirić-Matkowski contraction on X, take a point $x \in X$, and set $x_n = T^n x$ ($n \in \mathbb{N}$). Then, for every $\epsilon > 0$, there exist $\delta > 0$ such that $d(x_p, x_q) < \epsilon + \delta$ implies $d(x_{p+1}, x_{q+1}) \leq \epsilon$. By the above lemma,

Lemma 3.2. *Let $T : X \to X$ be a mapping. Suppose $d(T^n x, T^{n+1}x) \to 0$, for some $x \in X$. Then, for some $k \in \mathbb{N}$, either the picard iterates $T^n x$ $(n \geq k)$ are all different or they are all the same.*

Proof. Suppose $T^{k+m}x = T^k x$, for some $k, m \in \mathbb{N}$, and let m be the smallest positive integer with this property. If $m = 1$, that is $T^{k+1}x = T^k x$, then $T^n x = T^k x$, for $n \geq k$, and there is nothing to prove. If $m \geq 2$, then every two successive element in the following sequence are different:

$$T^k x, T^{k+1}x, \dots, T^{k+m-1}x, T^{k+m}x, T^{k+m+1}x, \dots$$

□

Theorem 3.3. *Let T be a self-map of X and $m(x, y)$ be a nonnegative function on $X \times X$. Suppose, for some point $x \in X$, the following conditions hold:*

(i) *for any $\epsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{Z}^+$ such that*

$$\forall p, q \geq N, \quad m(T^p x, T^q x) < \delta + \epsilon \Longrightarrow d(T^{p+1}x, T^{q+1}x) \leq \epsilon, \tag{3.3}$$

(ii) *condition (2.3) holds for any two subsequences $\{T^{p_i}x\}$ and $\{T^{q_i}x\}$ of $\{T^n x\}$,*

(iii) *$d(T^n x, T^{n+1}x) + d(T^n x, T^{n+2}x) \to 0$.*

Then $\{T^n x\}$ is a Cauchy sequence.

Proof. Using Lemma 3.1, condition (3.3) implies that, for every $\epsilon > 0$, for any two subsequences $\{T^{p_i}x\}$ and $\{T^{q_i}x\}$ of $\{T^n x\}$, if $\limsup m(T^{p_i}x, T^{q_i}x) \leq \epsilon$ then, for some N, $d(T^{p_i+1}x, T^{q_i+1}x) \leq \epsilon$ $(i \geq N)$. By Lemma 3.2, the Picard iterates $T^n x$ are eventually all the same, in which case $\{T^n x\}$ is obviously a Cauchy sequence, or they are all different. In the latter case, Theorem 2.3 shows that $\{T^n x\}$ is Cauchy. □

Corollary 3.4. *Let T be a Ćirić-Matkowski contraction on X. Then T has a unique fixed point z, and, moreover, for any $x \in X$, the sequence $\{T^n x\}$ converges to z in the strong sense.*

Proof. First, we show that T has at most one fixed point. Suppose $Tz = z$ and $y \neq z$. Then $d(Ty, Tz) = d(Ty, z) < d(y, z)$. Hence $Ty \neq y$.

Given $x \in X$, we consider the following two cases.

(a) There exists $k, m \in \mathbb{N}$ such that $T^{k+m}x = T^k x$.

(b) $T^n x$ $(n \in \mathbb{N})$ are all different.

In case (a), where $T^{k+m}x = T^k x$, for some $k, m \in \mathbb{N}$, we let m be the smallest positive integer with this property. If $m = 1$, that is $T^{k+1}x = T^k x$, then $T^n x = T^k x$, for $n \geq k$, and there is nothing to prove. If $m \geq 2$, then every two successive element in the following sequence are different:

$$T^k x, T^{k+1}x, \dots, T^{k+m-1}x, T^{k+m}x, T^{k+m+1}x, \dots$$

Recall that $x \neq y$ implies $d(Tx, Ty) < d(x, y)$. Hence

$$\begin{aligned} d(T^k x, T^{k+1}x) &= d(T^{k+m}x, T^{k+m+1}x) < d(T^{k+m-1}x, T^{k+m}x) \\ &< \cdots < d(T^{k+1}x, T^{k+2}x) < d(T^k x, T^{k+1}x). \end{aligned}$$

This is absurd.

In case (b), we let $x_n = T^n x$, and show that $d(x_n, x_{n+i}) \to 0$, for $i = 1, 2$. Since x_n $(n \in \mathbb{N})$ are all different, we have $d(x_{n+1}, x_{n+i+1}) < d(x_n, x_{n+i})$, for every n, that is, the sequence $\epsilon_n = d(x_n, x_{n+i})$ is decreasing and thus $\epsilon_n \downarrow \epsilon$ for some

$\epsilon \geq 0$. If $\epsilon > 0$, there is $\delta > 0$ such that $\epsilon_n = d(T^n x, T^{n+1}x) \leq \epsilon + \delta$ implies that $\epsilon_{n+1} = d(T^{n+1}x, T^{n+2}x) \leq \epsilon$. This is a contradiction since we have $\epsilon < \epsilon_n$, for all n. Hence, $d(x_n, x_{n+i}) \to 0$ $(i = 1, 2)$. Now, by Theorem 3.3, the sequence $\{T^n x\}$ is Cauchy. Since X is complete, $\{T^n x\}$ converges to some $z \in X$. By Proposition 1.4, we have

$$d(z, Tz) = \lim_{n\to\infty} d(T^n x, Tz) \leq \lim_{n\to\infty} d(T^{n-1}x, z) = 0.$$

Hence $Tz = z$, i.e., z is a fixed point of T. □

Lemma 3.5. *Let $\{x_n\}$ be a sequence in a ν-generalized metric space X such that x_n $(n \in \mathbb{N})$ are all different. If $d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \to 0$, then*

$$d(x_n, x_{n+m}) \to 0, \quad (m \geq 3).$$

Definiton 3.6. A self-mapping T of a ν-generalized metric space X is said to be *sequentially continuous* if $\{Tx_n\}$ converges to Tx whenever $\{x_n\}$ converges to x. The mapping T is called *asymptotically regular* if

$$d(T^n x, T^{n+1}x) + d(T^n x, T^{n+2}x) \to 0 \quad (x \in X).$$

We are now in a position to state and prove a version of Proinov's theorem, [11, Theorem 4.2], for ν-generalized metric spaces.

Theorem 3.7. *Let X be a complete ν-generalized metric space, and T be a sequentially continuous and asymptotically regular self-map of X. For $\gamma > 0$, define m on $X \times X$ by*

$$m(x, y) = d(x, y) + \gamma\big(d(x, Tx) + d(y, Ty)\big). \tag{3.4}$$

Suppose $d(Tx, Ty) < m(x, y)$, for every $x, y \in X$, with $x \neq y$, and, for any $\epsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}_0$ such that

$$\forall x, y \in X, \quad m(T^N x, T^N y) < \delta + \epsilon \Longrightarrow d(T^{N+1}x, T^{N+1}y) \leq \epsilon. \tag{3.5}$$

Then T has a unique fixed point z, and, for any $x \in X$, the Picard iterates $T^n x$ $(n \in \mathbb{N})$ converge to z in the strong sense.

Proof. First, let us prove that T has at most one fixed point. If $Ty = y$ and $Tz = z$. Then $m(y, z) = d(y, z) = d(Ty, Tz)$. Hence $y = z$.

Now, choose $x \in X$ and set $x_n = T^n x$ $(n \in \mathbb{N})$. Since T is assumed to be asymptotically regular, we have $d(x_n, x_{n+1}) \to 0$. Hence, (2.3) holds, for any two subsequences $\{x_{p_i}\}$ and $\{x_{q_i}\}$. By Theorem 2.3, the sequence $\{T^n x\}$ is Cauchy and, since X is complete, it converges to some point $z \in X$. Since T is sequentially continuous, we have $Tz = z$. □

References

[1] M. Abtahi, *Fixed point theorems for Meir-Keeler type contractions in metric spaces*, Fixed Point Theory (to appear).

[2] B. Alamri, T. Suzuki and L.A. Khan, *Caristi's Fixed Point Theorem and Subrahmanyam's Fixed Point Theorem in ν-Generalized Metric Spaces*, Journal of Function Spaces, 2015, Article ID 709391.

[3] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, Publicationes Mathematicae Debrecen, 57 (2000) 31–37.

[4] Lj. B. Ćirić, *A new fixed-point theorem for contractive mappings*, Publ. Inst. Math. (N.S) **30** (44) (1981), 25–27.

[5] Z. Kadelburg and S. Radenović, *On generalized metric spaces: A survey*, TWMS J. Pure Appl. Math., 5 (2014), 3–13.

[6] Z. Kadelburg and S. Radenović, *Fixed point results in generalized metric spaces without Hausdorff property*, Mathematical Sciences, vol. 8, article 125, 2014.
[7] R. Kannan, *Some results on fixed points-II*, Amer. Math. Monthly, 76 (1969), 405–408.
[8] W. A. Kirk and N. Shahzad, *Generalized metrics and Caristi's theorem*, Fixed Point Theory Appl., 2013, 2013:129.
[9] M. Kuczma, B. Choczewski, R. Ger, *Iterative functional equations, Encyclopedia of Mathematics and Applications*, vol. 32, Cambridge University Press, Cambridge, 1990.
[10] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969), 326–329.
[11] Petko D. Proinov, *Fixed point theorems in metric spaces*, Nonlinear Anal. 64 (2006), 546–557.
[12] B. Samet, *Discussion on 'a fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces' by A. Branciari*, Publicationes Mathematicae, vol. 76, no. 4, pp. 493–494, 2010.
[13] I.R. Sarma, J.M. Rao, S.S. Rao, *Contractions over generalized metric spaces*, Journal of Nonlinear Science and its Applications, vol. 2, no. 3, pp. 180–182, 2009.
[14] T. Suzuki, *Generalized metric spaces do not have the compatible topology*, Abstr. Appl. Anal., 2014, Art. ID 458098, 5 pp.
[15] T Suzuki, B Alamri and L.A. Khan, *Some notes on fixed point theorems in ν-generalized metric spaces*, Bull. Kyushu Inst. Tech. Pure Appl. Math. 62 (2015), 15–23.
[16] M. Turinici, *Functional contractions in local Branciari metric spaces*, ROMAI Journal, vol. 8, no. 2, pp. 189–2012.